Nonlinear stochastic equations with calculable steady states

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We consider generalizations of the Kardar-Parisi-Zhang equation that accommodate spatial anisotropies and the coupled evolution of several fields, and focus on their symmetries and nonperturbative properties. In particular, we derive generalized fluctuation-dissipation conditions on the form of the (nonlinear) equations for the realization of a Gaussian probability density of the fields in the steady state. For the amorphous growth of a single height field in one dimension we give a general class of equations with exactly calculable (Gaussian and more complicated) steady states. In two dimensions, we show that any anisotropic system evolves in long time and length scales either to the usual isotropic strong coupling regime or to a linearlike fixed point associated with a hidden symmetry. Similar results are derived for textural growth equations that couple the height field with additional order parameters which fluctuate on the growing surface. In this context, we propose phenomenological equations for the growth of a crystalline material, where the height field interacts with lattice distortions, and identify two special cases that obtain Gaussian steady states. In the first case compression modes influence growth and are advected by height fluctuations, while in the second case it is the density of dislocations that couples with the height.

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I. INTRODUCTION

Nonlinear stochastic partial differential equations appear extensively in problems of equilibrium and nonequilibrium statistical physics. For systems in thermal equilibrium, the form of these equations is constrained by fluctuation-dissipation conditions [1] that ensure convergence of the steady-state probability distribution to the appropriate Boltzmann weight [2]. Nonequilibrium systems are not similarly constrained, and there is no simple way of finding their behavior in steady state (if any). However, there are examples in which steady states can be found exactly as solutions of the associated Fokker-Planck equations. In this paper we review some such examples, and introduce several new ones, along the way seeking general principles for finding steady states associated to nonlinear stochastic equations exactly.

The simplest equation, which serves as the prototype for our investigations, is the Kardar-Parisi-Zhang (KPZ) equation [3]

$$\partial_t h = \nu \nabla^2 h + \frac{1}{2}\lambda (\boldsymbol{\nabla} h)^2 + \eta, \qquad (1)$$

describing the nonequilibrium fluctuations of a (height) field $h(\mathbf{x},t)$. The equation is equivalent to the Burgers equation (for a field $\mathbf{u} = \boldsymbol{\nabla} h$) for vorticity-free turbulence [4], and appears in various guises in the study of domain walls [5] and directed polymers [6,7] in a random potential, surface growth [8], and even the gene (or protein) sequence alignment problem [9,10]. Still more problems can be formulated as generalizations of the KPZ equation that accommodate spatial anisotropy or the interplay of several fields. Examples in the literature include the dynamics of a vicinal surface [11,12], the growth of two coupled surfaces [13] or of a magnetic surface [14–16], and the transport of a flux line or polymer [17].

The stochastic aspect of Eq. (1) is due to the noise $\eta(\mathbf{x},t)$, which has zero mean and short-range correlations in space and time. In the absence of the nonlinear term proportional to λ, it reduces to a standard Langevin equation, with a Gaussian steady state. In one dimension, the nonlinear term does not modify this steady state as the associated probability current in the Fokker-Planck equation is zero. This observation motivates our search for other equations with this property, namely an easily guessed (equilibrium) steady state which is not affected by the additional (nonequilibrium) nonlinearities.

In Sec. II, we start by constructing the Fokker-Planck equation for the one-dimensional KPZ equation and explicitly showing that the probability current due to the nonlinear term does not modify the steady state, as it appears in an integral of a complete derivative. This observation is then used as a basis for constructing other one-dimensional nonlinear equations that share this property. Indeed, we find that the class of such equations is quite large, including some equations already considered in the literature.

Higher-dimensional versions of the KPZ equation may also obtain a Gaussian steady state in spite of their nonlinear character. We discuss such a case in Sec. III, namely, an anisotropic variant of the KPZ equation in two dimensions with nonlinear terms of opposite signs in the two directions. Using renormalization group methods, Wolf [12] showed that this model indeed flows under renormalization to a linear fixed point. Generalizations of this equation with calculable steady states are also constructed; they all share a hidden symmetry under reflection, absent in the isotropic KPZ equation.

The examples from one and two dimensions motivate the search for more general principles governing the existence of simple steady states, taken up in Sec. IV. Specifically, we consider stochastic dynamics of multiple fields coupled by nonlinear (possibly anisotropic) generalizations of the KPZ term, and ask whether they admit Gaussian steady states. A direct solution of the Fokker-Planck equation becomes considerably more difficult and, instead, we derive two sets of general prescriptions on the coefficients for this to occur. These prescriptions may be viewed as generalized fluctuation-dissipation relations [18] and are quite restrictive. In particular, they cannot be satisfied in three and higher dimensions as we show in Sec. V.

Having obtained general prescriptions, in Sec. VI we apply them to equations for coupled fields in one and two dimensions. Some of the examples we discuss correspond to equations that have already appeared in the literature, in particular, pertaining to the dynamics of a flux line or polymer (Sec. VI A) and to the growth of a magnetic film (Sec. VI B). However, in Sec. VI C we propose a set of equations to describe the coupling of the strain field of a growing crystal to its height fluctuations. We find that Gaussian steady states are indeed permitted for these equations in special cases.

The Appendix treats a simple example aimed at illustrating how the systematic approach may be extended to exactly calculable non-Gaussian steady states.

II. GENERALIZED GROWTH EQUATIONS IN ONE DIMENSIONS

Consider the probability distribution $\mathcal{P}[h]$ for configurations of the field $h(\mathbf{x})$. As the surface changes in time according to Eq. (1), the corresponding probability evolves according to the Fokker-Planck equation

$$\partial_t \mathcal{P} = \int d^d x \left\{ -\left[\nu \nabla^2 h + \frac{1}{2}\lambda (\nabla h)^2 \right] \frac{\delta \mathcal{P}}{\delta h} + D \frac{\delta^2 \mathcal{P}}{\delta h^2} \right\}. \quad (2)$$

The term in the square brackets is due to the deterministic probability current and the remainder comes from the stochastic noise, assumed Gaussian with $\langle \eta(\mathbf{x},t) \rangle = 0$ and

$$\langle \eta(\mathbf{x},t)\, \eta(\mathbf{x}',t') \rangle = 2D\, \delta^d(\mathbf{x}-\mathbf{x}')\, \delta(t-t'). \quad (3)$$

In equilibrium, $D = k_B T$; more generally, D is a measure of the magnitude of the noise.

A steady-state solution is one for which $\partial_t \mathcal{P} = 0$. In the absence of the nonlinear term, the steady-state solution of Eq. (2) is a simple Gaussian,

$$\mathcal{P} = N \exp\left(-\frac{\nu}{2D} \int d^d x (\nabla h)^2 \right), \quad (4)$$

where N is a normalization constant. In general, this is not a steady state for $\lambda \neq 0$. In one dimension, however, the contribution of the nonlinear term to the probability current can be simplified to

$$\partial_t \mathcal{P} = -\mathcal{P} \int dx \frac{\nu\lambda}{2D} (\partial_x h)^2 \partial_{xx} h = -\mathcal{P} \int dx\, \partial_x \left[\frac{\nu\lambda}{6D} (\partial_x h)^3 \right], \quad (5)$$

a surface integral safely set to zero in the limit of an infinite system. Thus, the steady-state spatial correlations are not influenced by the presence of the KPZ nonlinearity and coincide with the Gaussian (Edwards-Wilkinson [19]) ones. We make no statements about the stability of the steady state of Eq. (4); however, here, and in every other example for which simulations are available, numerical results indicate that the simple Gaussian steady states we discuss are indeed the ones achieved at long time.

The one-dimensional KPZ equation is a particular instance of a more general class of equations

$$\partial_t h = f(\partial_x h) \partial_x \partial_x h + g(\partial_x h) + \eta, \quad (6)$$

with f and g arbitrary functions. These equations obtain a steady state

$$\mathcal{P} = N \exp\left(-\frac{1}{D} \int dx F(\partial_x h) \right), \quad (7)$$

where f is the second derivative of F [i.e., $d^2F(u)/du^2 = f(u)$]. For $g(\partial_x h) = 0$, Eq. (6) is again a standard Langevin equation, while the contribution of this function to the probability current is

$$\partial_t \mathcal{P} = -\mathcal{P} \int dx \frac{1}{D} g(\partial_x h) f(\partial_x h) \partial_x \partial_x h$$
$$= -\mathcal{P} \int dx\, \partial_x \left[\frac{\nu}{D} G(\partial_x h) \right] = 0, \quad (8)$$

where G is the primitive of gf [i.e., $dG(u)/du = g(u)f(u)$].

The special case of a cubic nonlinearity $g(\partial_x h) = \frac{1}{2}\lambda(\partial_x h)^2 + \frac{1}{6}\lambda'(\partial_x h)^3$ (with $f=1$) was introduced to describe an interface separating stationary phases of the Toom model [20]. The Gaussian steady state corroborates the marginal irrelevance [21] of the cubic term, and implies spatial correlations of the form

$$\langle [h(x) - h(x')]^2 \rangle^{1/2} \sim |x - x'|^{1/2}. \quad (9)$$

III. ANISOTROPIC EQUATIONS IN TWO DIMENSIONS

To describe the growth of a vicinal (slightly miscut from a low index facet) surface, Villain introduced [11] an anisotropic version of the KPZ equation, which was subsequently studied with a renormalization group calculation by Wolf [12]. This generalized equation has the form

$$\partial_t h = \nu_x \partial_x^2 h + \nu_y \partial_y^2 h + \tfrac{1}{2}\lambda_x (\partial_x h)^2 + \tfrac{1}{2}\lambda_y (\partial_y h)^2 + \eta. \quad (10)$$

Under renormalization, the subspace with $\nu_x/\nu_y = \lambda_x/\lambda_y$ is fixed and equivalent to the isotropic KPZ equation modulo a rescaling of x or y. This subspace is locally attractive, so that the equation flows to a strong-coupling limit if λ_x and λ_y have the same sign (stability requires $\nu_x, \nu_y \geqslant 0$). The more surprising behavior arises when the product $\lambda_x \lambda_y$ is negative, in which case the flows converge to a fixed point with vanishing nonlinearities.

This vanishing of nonlinearities at long length and time scales suggests a Gaussian steady-state probability density, as corroborated by an exact solution of a discrete model belonging to the same universality class [22] and by a direct solution of the Fokker-Planck equation [23]. Indeed, the *Ansatz*

$$\mathcal{P}=N\exp\left(-\frac{1}{2D}\int dxdy[\nu_x(\partial_x h)^2+\nu_y(\partial_y h)^2]\right), \quad (11)$$

with the generalized fluctuation-dissipation condition $\nu_x/\nu_y = -\lambda_x/\lambda_y$, solves for the steady state. To verify this, we note that the contributions from the nonlinearities take the form

$$\begin{aligned}\partial_t\mathcal{P}=&-\frac{\mathcal{P}}{2D}\int dxdy[\lambda_x(\partial_x h)^2+\lambda_y(\partial_y h)^2](\nu_x\partial_x^2 h+\nu_y\partial_y^2 h)\\ =&-\frac{\mathcal{P}}{2D}\int dxdy\left\{\partial_x\left[\frac{\lambda_x\nu_x}{3}(\partial_x h)^3+\lambda_y\nu_x(\partial_x h)(\partial_y h)^2\right]\right.\\ &+\partial_y\left[\frac{\lambda_y\nu_y}{3}(\partial_y h)^3+\lambda_x\nu_y(\partial_x h)^2(\partial_y h)\right]\\ &\left.+2\partial_x h\partial_y h\partial_x\partial_y h(\lambda_y\nu_x+\lambda_x\nu_y)\right\}. \quad (12)\end{aligned}$$

If $\lambda_y\nu_x+\lambda_x\nu_y=0$, the above contribution is the divergence of a vector field, and hence vanishes subject to the usual boundary conditions. This nonperturbative derivation complements the renormalization group analysis [12] which captures perturbatively the character of this state at large scales and the dynamics that lead to it. In particular, it demonstrates that a Gaussian steady state (and the logarithmic roughness it implies) obtains at *any* length scale, and not only in the long wavelength limit.

By suitable rescalings of x and y, we can make $\nu_x=\nu_y=\nu$, so that the steady state reduces to Eq. (4) with $d=2$. Trivially, this steady state also holds for any equation related to

$$\partial_t h=\nu\nabla^2 h+\frac{\lambda}{2}[(\partial_x h)^2-(\partial_y h)^2]+\eta, \quad (13)$$

by a rotation of the plane. This class comprises all the equations of the form

$$\partial_t h=\nu\nabla^2 h+\frac{\lambda_1}{2}[(\partial_x h)^2-(\partial_y h)^2]+\lambda_2\partial_x h\partial_y h+\eta, \quad (14)$$

where arctan(λ_2/λ_1)/2 is the plane rotation angle; in particular, the equation

$$\partial_t h=\nu\nabla^2 h+\lambda\partial_x h\partial_y h+\eta \quad (15)$$

is obtained from Eq. (13) by a 45° rotation.

The surprisingly simple steady state of Eq. (13) results from a " hidden" symmetry under the transformation

$$h\to -h,\quad x\to y,\quad y\to x. \quad (16)$$

The symmetry $h\to -h$ is precisely the one broken by the isotropic KPZ nonlinearity. It is restored here, provided the plane is also inverted about an appropriate axis: the bisector in the case of Eq. (13), the x or y axis in the case of Eq. (15), and a properly rotated axis in the general case of Eq. (14). This hidden symmetry sheds light on the renormalization group analysis [12], as any two-dimensional anisotropic KPZ term may be written, upon rotation of the plane, as the sum of an isotropic part and the antisymmetric part of Eq. (13) whose subspace is invariant under renormalization.

By analogy to Eq. (6), we can generalize Eq. (15) to include a more complicated Laplacian term as

$$\partial_t h=f_x(\partial_x h)\partial_x^2 h+f_y(\partial_y h)\partial_y^2 h+\lambda\partial_x h\partial_y h+\eta. \quad (17)$$

Indeed, it is easy to check that the probability density

$$\mathcal{P}=N\exp\left(-\frac{1}{D}\int dxdy[F_x(\partial_x h)+F_y(\partial_y h)]\right), \quad (18)$$

where $f_x(u)=dF_x(u)/du$ and $f_y(u)=dF_y(u)/du$, is stationary.

IV. GENERAL PRESCRIPTIONS FOR GAUSSIAN STEADY STATES

The solution of Fokker-Planck equations by direct check of *Ansätze* soon becomes laborious beyond simple one- and two-dimensional examples. Instead, we derive general prescriptions on the structure of the equations of evolution, for the realization of Gaussian steady states. We consider equations of the form

$$\partial_t h_i(\mathbf{x},t)=\mathcal{L}_{\mathbf{x}}^{(i)}[h]+\mathcal{N}_{\mathbf{x}}^{(i)}[h]+\eta_i(\mathbf{x},t), \quad (19)$$

for n coupled fields ($i=1,\ldots,n$), $\mathbf{x}\in\mathbb{R}^d$, and Gaussian (thermal) noise with correlator

$$\langle\eta_i(\mathbf{x},t)\eta_j(\mathbf{x}',t')\rangle=2D_i\delta_{ij}\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t'). \quad (20)$$

$\mathcal{L}_{\mathbf{x}}$ and $\mathcal{N}_{\mathbf{x}}$ denote linear and nonlinear functionals of the fields, respectively, evaluated at point $\mathbf{x}$. If η_i and $\eta_{j\neq i}$ are uncorrelated [24], the Fokker-Planck equation reads

$$\partial_t\mathcal{P}=\int d^dx\sum_i\frac{\delta}{\delta h_i}\left[-(\mathcal{L}_{\mathbf{x}}^{(i)}+\mathcal{N}_{\mathbf{x}}^{(i)})\mathcal{P}+\frac{\delta\mathcal{P}}{\delta h_i}\right], \quad (21)$$

where we have absorbed D_i in a rescaling of h_i (by $\sqrt{D_i}$), and reduces to

$$\partial_t\mathcal{P}=\int d^dx\sum_i\left[-(\mathcal{L}_{\mathbf{x}}^{(i)}+\mathcal{N}_{\mathbf{x}}^{(i)})\frac{\delta\mathcal{P}}{\delta h_i}+\frac{\delta^2\mathcal{P}}{\delta h_i^2}\right], \quad (22)$$

if $\mathcal{L}$ and $\mathcal{N}$ depend upon the derivatives of h only. We are looking for a Gaussian probability density

$$\mathcal{P}=Ne^{-Q[h]}, \tag{23}$$

with $Q[h]$ a quadratic form and N a normalizing factor, that solves the steady state $\partial_t\mathcal{P}=0$, i.e.,

$$\int d^dx\sum_i\left[-(\mathcal{L}_{\mathbf{x}}^{(i)}+\mathcal{N}_{\mathbf{x}}^{(i)})\left(-\frac{\delta Q}{\delta h_i}\right)+\left(\frac{\delta Q}{\delta h_i}\right)^2\right]=0. \tag{24}$$

The quadratic terms cancel if $\mathcal{L}$ and Q are related through

$$\mathcal{L}_{\mathbf{x}}^{(i)}=-\frac{\delta Q}{\delta h_i}, \tag{25}$$

and it remains to find the form of $\mathcal{L}$ and $\mathcal{N}$ for which the integral

$$\mathcal{J}[h]\equiv\int d^dx\sum_i \mathcal{N}_{\mathbf{x}}^{(i)}\frac{\delta Q}{\delta h_i}=-\int d^dx\sum_i \mathcal{N}_{\mathbf{x}}^{(i)}\mathcal{L}_{\mathbf{x}}^{(i)} \tag{26}$$

vanishes. This is the case if the integrand either vanishes identically or is the divergence of a vector field. In either case, the integrand is unchanged by a variation $\delta h(\mathbf{x})$ that vanishes at infinity. If δh is localized at $\mathbf{x}$, the condition $\mathcal{J}[h+\delta h]=J[h]$ translates, to first order in δh, into

$$\frac{\delta}{\delta h_j(\mathbf{x})}\int d^dy\mathcal{N}_{\mathbf{y}}^{(i)}\mathcal{L}_{\mathbf{y}}^{(i)}=0, \tag{27}$$

where the summation over i is understood.

For coupled KPZ-like equations [25] of the form

$$\partial_t h_i=\nu_{ij\alpha\beta}\partial_\alpha\partial_\beta h+\tfrac{1}{2}\lambda_{ijk\alpha\beta}\partial_\alpha h_j\partial_\beta h_k+\eta_i, \tag{28}$$

where latin indices $i,j,k=1,\ldots,n$ refer to field components and greek indices $\alpha,\beta=1,\ldots,d$ refer to spatial components, Eq. (27) reduces to

$$\begin{aligned}&\nu_{ij\alpha\beta}\lambda_{ikl\gamma\delta}(\partial_\alpha\partial_\beta\partial_\gamma h_k\partial_\delta h_l+\partial_\alpha\partial_\gamma h_k\partial_\beta\partial_\delta h_l)\\&\quad-\nu_{ik\alpha\beta}\lambda_{ijl\gamma\delta}(\partial_\alpha\partial_\beta\partial_\gamma h_k\partial_\delta h_l+\partial_\alpha\partial_\beta h_k\partial_\gamma\partial_\delta h_l)=0,\end{aligned} \tag{29}$$

after some renaming of the indices and using the fact that ν and λ can always be chosen to satisfy the equalities

$$\nu_{ij\alpha\beta}=\nu_{ij\beta\alpha}\quad\text{and}\quad\lambda_{ijk\alpha\beta}=\lambda_{ikj\beta\alpha}. \tag{30}$$

Furthermore, Eq. (25) requires the symmetry

$$\nu_{ij\alpha\beta}=\nu_{ji\alpha\beta}, \tag{31}$$

and the stationary probability density reads

$$\mathcal{P}=N\exp\left(-\int d^dx\frac{\nu_{ij\alpha\beta}}{2}\partial_\alpha h_i\partial_\beta h_j\right). \tag{32}$$

Since repeated indices are summed over, Eq. (29) represents n conditions—one for each possible value of the index j. Each of these in fact encapsulates more than one constraint: Eq. (29) is composed of terms that come in one of two derivative structures and, as the equation must be true for arbitrary h, terms of a given derivative structure must cancel independently. This is ensured by the following two sets of *generalized fluctuation-dissipation conditions* on the tensors ν and λ: The first condition (I) comes from grouping terms in Eq. (29) which are products of first and third derivatives (such as $\partial_\alpha\partial_\beta\partial_\gamma h_k\partial_\delta h_l$), and reads

$$\sum_P(\nu_{ijP(\alpha)P(\beta)}\lambda_{iklP(\gamma)\delta}-\nu_{ikP(\alpha)P(\beta)}\lambda_{ijlP(\gamma)\delta})=0, \tag{33}$$

where the summation runs over the six permutations of the indices α, β, γ. A second condition (II) comes from grouping terms of the form $\partial_\alpha\partial_\gamma h_k\partial_\beta\partial_\delta h_l$, and gives

$$\begin{aligned}\sum_{R,R'}&(2\nu_{ijR(\alpha)R'(\gamma)}\lambda_{iklR(\beta)R'(\delta)}-\nu_{ikR(\alpha)R(\beta)}\lambda_{ijlR'(\gamma)R'(\delta)}\\&-\nu_{ilR'(\gamma)R'(\delta)}\lambda_{ijkR(\alpha)R(\beta)})=0,\end{aligned} \tag{34}$$

where the summation runs over the two permutations of the indices α, β and the two permutations of the indices γ, δ. Each pair of conditions corresponds to a given choice of numerical values for j, k, l, α, β, γ, and δ.

Conditions (I) and (II) are *necessary* for a Gaussian solution of the steady-state Fokker-Planck equation. They are also *sufficient* conditions, since $\mathcal{J}[h+\delta h]=\mathcal{J}[h]$ to first order *for any* δh implies $\mathcal{J}[h+\delta h]=\mathcal{J}[h]$ to all orders, and $\mathcal{J}[h]=\text{const}\equiv c$. But normalization of $\mathcal{P}$ allows only $c=0$ (otherwise $\mathcal{P}$ either increases or decreases uniformly), and consequently $\partial_t\mathcal{P}=0$.

V. ABSENCE OF GAUSSIAN STEADY STATES IN THREE AND HIGHER DIMENSIONS

If the matrix of Laplacian coefficients $\nu_{ij\alpha\beta}$ is positive definite, as required by infrared stability, it can be diagonalized into

$$\nu_{ij\alpha\beta}=\nu_{ij}\delta_{\alpha\beta}, \tag{35}$$

by successive rotations and rescalings, and prescriptions (I) and (II) simplify correspondingly. In three and higher dimensions we may choose the space indices, in applying the prescriptions, such that $\alpha=\gamma$ while $\alpha\neq\beta\neq\delta\neq\alpha$. With this choice, it is straightforward to check that prescription (II) forbids a nonvanishing contraction of tensors with different space indices, hence enforces the partially diagonal form

$$\nu_{ij}\lambda_{ikl\alpha\beta}\equiv\nu_{ij}\lambda_{ikl\alpha\alpha}\delta_{\alpha\beta} \tag{36}$$

(where i is summed over but α is not). With this constraint, prescription (II) takes the form

$$\begin{aligned}\sum_{R,R'}&[\nu_{ij}\lambda_{jklR(\beta)R(\beta)}\delta_{R(\alpha)R'(\gamma)}\delta_{R(\beta)R'(\delta)}-\tfrac{1}{2}(\nu_{ik}\lambda_{ijlR'(\gamma)R'(\gamma)}\\&+\nu_{il}\lambda_{ijkR(\alpha)R(\alpha)})\delta_{R(\alpha)R(\beta)}\delta_{R'(\gamma)R'(\delta)}]=0.\end{aligned} \tag{37}$$

This equation expresses a set of different conditions, one for each choice of values of the indices that are not summed over. Specifically, for a particular choice in which $\alpha=\gamma \neq \beta=\delta$, Eq. (37) translates into

$$\nu_{ij}\lambda_{ikl\alpha\alpha}=-\nu_{ij}\lambda_{ikl\beta\beta}. \tag{38}$$

For the sake of visual ease, let us define the object $\varphi_\alpha \equiv \nu_{ij}\lambda_{ikl\alpha\alpha}$ (the dependence of φ_α on j, k, l is tacit), in terms of which this identity reads

$$\varphi_\alpha=-\varphi_\beta; \tag{39}$$

clearly, if it is possible to choose three or more distinct values of the indices α, β, this condition is frustrated and admits only the trivial solution

$$\varphi_\alpha=\nu_{ij}\lambda_{ikl\alpha\alpha}=0, \tag{40}$$

for all j, k, l, α. Viewed as a vector identity, this requires that any vector $\boldsymbol{\nu}_j$ be orthogonal to any vector $\boldsymbol{\lambda}_{kl\alpha\alpha}$ (with components labeled by $i=1,\ldots,n$). As long as the matrix ν_{ij} is nondegenerate, there are n nonvanishing independent vectors $\boldsymbol{\nu}_j$ (for $j=1,\ldots,n$) and Eq. (40) is satisfied only if the vectors $\boldsymbol{\lambda}_{kl\alpha\alpha}$ vanish for all k, l, α. Hence, no Gaussian steady state is achievable in three and higher dimensions if nonlinearities are present in the equations of evolution [26].

VI. COUPLED FIELDS IN ONE AND TWO DIMENSIONS

In the case of a single field fluctuating in two dimensions, prescriptions (I) and (II) immediately enforce the form of Eq. (14) for which a Gaussian steady state may be reached, as we checked explicitly in Sec. III. In what follows, we discuss examples in which the coupling among several fluctuating fields broadens the class of nonlinear equations with Gaussian steady states beyond this specific anisotropic form (with coefficients of opposite signs).

A. Coupled lines and drifting polymers

An array of fluctuating directed lines [5–7] is parametrized by a single variable, consequently the greek indices in Eq. (28) all take the same value and may be omitted. Equation (28) then reduces to

$$\partial_t h_i=\nu_{ij}\partial_x\partial_x h_j+\tfrac{1}{2}\lambda_{ijk}\partial_x h_j\partial_x h_k+\eta_i, \tag{41}$$

a generalization of the one-dimensional KPZ equation for several coupled fields. In this simple case, prescriptions (I) and (II) are fulfilled by any tensors ν, λ such that

$$\nu_{ij}\lambda_{ikl}=\nu_{ik}\lambda_{ijl}=\nu_{il}\lambda_{ikj}, \tag{42}$$

where λ_{ijk} always can be chosen symmetric in j, k, and the sum over i is understood. It is easy to check that these relations ensure a stationary probability density

$$\mathcal{P}=N\exp\left(-\int dx\,\frac{\nu_{ij}}{2}\partial_x h_i\partial_x h_j\right). \tag{43}$$

Indeed, with this *Ansatz*

$$\begin{aligned}\partial_t\mathcal{P}&=-\int dx\,\frac{1}{2}\lambda_{ikl}\partial_x h_k\partial_x h_l\nu_{ij}\partial_x\partial_x h_j\mathcal{P}\\&=-\int dx\,\frac{1}{6}(\nu_{ij}\lambda_{ikl}\partial_x\partial_x h_j\partial_x h_k\partial_x h_l\\&\quad+\nu_{ik}\lambda_{ijl}\partial_x h_j\partial_x\partial_x h_k\partial_x h_l+\nu_{il}\lambda_{ikj}\partial_x h_j\partial_x h_k\partial_x\partial_x h_l)\mathcal{P},\end{aligned} \tag{44}$$

where we have renamed the mute indices to obtain the last equality. But Eq. (42) implies that all three coefficients are identical, and

$$\partial_t\mathcal{P}=-\int dx\,\frac{1}{6}\nu_{ij}\lambda_{ikl}\partial_x(\partial_x h_j\partial_x h_k\partial_x h_l)=0. \tag{45}$$

A special case of Eq. (41),

$$\begin{aligned}\partial_t h_\parallel&=\nu_\parallel\partial_x\partial_x h_\parallel+\frac{1}{2}\lambda_\parallel(\partial_x h_\parallel)^2+\frac{1}{2}\lambda_\perp(\partial_x h_\perp)^2+\eta_\parallel,\\\partial_t h_\perp&=\nu_\perp\partial_x\partial_x h_\perp+\lambda_\times\partial_x h_\parallel\partial_x h_\perp+\eta_\perp\end{aligned} \tag{46}$$

was introduced in Refs. [17] to describe a directed polymer drifting perpendicularly to itself. Here $h_\parallel$ and $h_\perp$ are interpreted not as height fields associated with two different lines embedded in two dimensions, but rather as dynamically coupled longitudinal and transverse (to the average velocity of the polymer) fluctuations of a single line embedded in three dimensions. The conditions of Eq. (42) for a Gaussian steady state simplify to

$$\nu_\parallel\lambda_\perp=\nu_\perp\lambda_\times, \tag{47}$$

in agreement with a direct check [17,27].

In the simplest case with identical longitudinal and transverse coefficients, a stationary Gaussian distribution follows trivially from the steady-state properties of the one-dimensional KPZ equation (discussed in Sec. II), since the equations

$$\begin{aligned}\partial_t h_\parallel&=\nu\partial_x\partial_x h_\parallel+\frac{1}{2}\lambda[(\partial_x h_\parallel)^2+(\partial_x h_\perp)^2]+\eta_\parallel,\\\partial_t h_\perp&=\nu\partial_x\partial_x h_\perp+\lambda\partial_x h_\parallel\partial_x h_\perp+\eta_\perp\end{aligned} \tag{48}$$

are equivalent to

$$\begin{aligned}\partial_t h_+&=\nu\partial_x\partial_x h_++\frac{1}{2}\lambda(\partial_x h_+)^2+\eta_+,\\\partial_t h_-&=\nu\partial_x\partial_x h_-+\frac{1}{2}\lambda(\partial_x h_-)^2+\eta_-,\end{aligned} \tag{49}$$

with $h_\pm=h_\parallel\pm h_\perp$ and $\eta_\pm=\eta_\parallel\pm\eta_\perp$. Clearly, the remark extends to higher spatial dimensions, where

$$\partial_t h_\parallel = \nu\nabla^2 h_\parallel + \frac{1}{2}\lambda[(\nabla h_\parallel)^2 + (\nabla h_\perp)^2] + \eta_\parallel ,$$
$$\partial_t h_\perp = \nu\nabla^2 h_\perp + \lambda \nabla h_\parallel \cdot \nabla h_\perp + \eta_\perp \tag{50}$$

transform into

$$\partial_t h_+ = \nu\nabla^2 h + \frac{1}{2}\lambda(\nabla h_+)^2 + \eta_+ ,$$
$$\partial_t h_+ = \nu\nabla^2 h_- + \frac{1}{2}\lambda(\nabla h_-)^2 + \eta_- . \tag{51}$$

Therefore, the " roughness" exponent $\zeta_\perp$ of a passive scalar $h_\perp$ advected [28,29] by a Burgers flow $\mathbf{u} = \nabla h_\parallel$, defined through

$$\langle [h_\perp(\mathbf{x}) - h_\perp(\mathbf{y})]^2 \rangle^{1/2} \sim |\mathbf{x}-\mathbf{y}|^{\zeta_\perp}, \tag{52}$$

is none other than the KPZ roughness exponent [3].

B. Magnetic growth

In a growing magnetic material, the spins may be assumed frozen in the bulk while still fluctuating on the surface, which itself fluctuates in height [14,15]. For the case of XY spins, described by a single angular field $\theta(x,y,t)$, Ref. [16] notes that a two-dimensional version of Eqs. (46) governs these nonequilibrium coupled fluctuations. In the modified notation, these equations of evolution read

$$\partial_t h = \nu_h \nabla^2 h + \frac{1}{2}\lambda_{hh}(\nabla h)^2 + \frac{1}{2}\lambda_{\theta\theta}(\nabla\theta)^2 + \eta_h ,$$
$$\partial_t \theta = \nu_\theta \nabla^2 \theta + \lambda_{h\theta} \nabla h \cdot \nabla\theta + \eta_\theta . \tag{53}$$

(Obviously, Eqs. (53) fail to capture the periodic nature of θ, and with it the potentially relevant presence of topological defects.)

Using prescriptions (I) and (II), it is easy to show that no Gaussian steady state exists for such isotropic equations as long as h and θ are decoupled at the linear level. However, the stationary Gaussian distribution

$$\mathcal{P} = N\exp\left(-\int dxdy \left[\frac{\nu_h}{2}(\nabla h)^2 + \frac{\nu_\theta}{2}(\nabla\theta)^2 \right] \right), \tag{54}$$

is achieved by a natural extension of the anisotropic Eq. (13),

$$\partial_t h = \nu_h \nabla^2 h + \frac{1}{2}\lambda_{hh}[(\partial_x h)^2 - (\partial_y h)^2] + \frac{1}{2}\lambda_{\theta\theta}[(\partial_x\theta)^2 - (\partial_y\theta)^2] + \eta_h ,$$
$$\partial_t \theta = \nu_\theta \nabla^2\theta + \lambda_{h\theta}(\partial_x h \partial_x\theta - \partial_y h \partial_y \theta) + \eta_\theta , \tag{55}$$

provided $\nu_h \lambda_{\theta\theta} = \nu_\theta \lambda_{h\theta}$ [in direct analogy to Eq. (47)]. We note also that these equations again satisfy the symmetry of Eq. (16).

C. Crystalline growth

The height fluctuations of a material characterized by internal order parameters, as in the above case of a growing XY magnet, are subjected to the fluctuations of these order parameters. Conversely, the evolution of the internal order parameters depends on the height fluctuations. In contrast to amorphous growth, we may say that the fluctuations of an ordered material results from a *textural growth*, as the additional fields invest the interface with a texture that constrains its fluctuations. The prime example is that of the growth of a crystal in which surface phonons interact with height fluctuations. In analogy with Eqs. (53), we propose the following equations for isotropic crystalline growth:

$$\partial_t h = \nu_h \nabla^2 h + \frac{1}{2}\lambda_{hh}(\nabla h)^2 + \frac{1}{2}\lambda_{uu}^{(1)}(\nabla\cdot\mathbf{u})^2 + \frac{1}{2}\lambda_{uu}^{(2)}\nabla u_i \cdot \nabla u_i + \frac{1}{2}\lambda_{uu}^{(3)}\partial_i \mathbf{u}\cdot\nabla u_i + \eta_h ,$$
$$\partial_t u_i = \nu_u^{(1)}\nabla^2 u_i + \nu_u^{(2)}\partial_i \nabla\cdot\mathbf{u} + \lambda_{hu}^{(1)}\partial_i h \nabla\cdot\mathbf{u} + \lambda_{hu}^{(2)}\nabla h\cdot\nabla u_i + \lambda_{hu}^{(3)}\nabla h \cdot \partial_i \mathbf{u} + \eta_i , \tag{56}$$

where $\mathbf{u}(x,y,t)$ is the surface displacement vector field, and $\nu_u^{(1)}$, $\nu_u^{(2)}$ are related to the usual Lamé coefficients through

$$\nu_u^{(1)} = \mu_{\text{Lamé}} ,$$
$$\nu_u^{(2)} = \mu_{\text{Lamé}} + \lambda_{\text{Lamé}} . \tag{57}$$

One would like to know, given the richness of Eqs. (56), what phases they describe beyond the usual KPZ (amorphous) phase. As a first step towards a complete answer, we discuss crystalline growth equations that admit an exact Gaussian steady state.

Equations (56) tacitly assume a triangular lattice at the microscopic level, since other lattices would be reflected in an anisotropic continuum limit that reproduces the appropriate crystal symmetries. In the context of anisotropic equations, trivial extensions of Eqs. (55) support a Gaussian steady state.

Rather than dwelling on these examples, we turn to a new possibility, namely, isotropic equations of the form of Eqs. (56) that admit a stationary Gaussian probability density, as allowed by the presence of linear couplings between u_x and u_y. A somewhat tedious but straightforward application of prescriptions (I) and (II) to Eqs. (56) yields two (and only two) nontrivial solutions, characterized by a vanishing shear modulus $\nu_u^{(1)} = 0$ and a vanishing bulk modulus $\nu_u^{(1)} + \nu_u^{(2)} = 0$.

With a vanishing shear modulus $\nu_u^{(1)} = 0$, prescriptions (I) and (II) imply

$$\lambda_{hh} = \lambda_{uu}^{(2)} = \lambda_{uu}^{(3)} = \lambda_{hu}^{(2)} = \lambda_{hu}^{(3)} = 0 \tag{58}$$

and

$$\nu_h \lambda_{uu}^{(1)} = \nu_u^{(2)} \lambda_{hu}^{(1)} , \tag{59}$$

resulting in the equations of motion

$$\partial_t h=\nu_h\nabla^2 h+\frac{1}{2}\lambda_{uu}(\boldsymbol{\nabla}\cdot\mathbf{u})^2+\eta_h,$$
$$\partial_t\mathbf{u}=\nu_u\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})+\lambda_{hu}\boldsymbol{\nabla}h(\boldsymbol{\nabla}\cdot\mathbf{u})+\boldsymbol{\eta},\tag{60}$$

where the coefficients have been renamed ($\nu_u\equiv\nu_u^{(2)}$, $\lambda_{uu}\equiv\lambda_{uu}^{(1)}$, $\lambda_{hu}\equiv\lambda_{hu}^{(1)}$). The probability density

$$\mathcal{P}=N\exp\left(-\int dxdy\left[\frac{\nu_u}{2}(\boldsymbol{\nabla}h)^2+\frac{\nu_u}{2}(\boldsymbol{\nabla}\cdot\mathbf{u})^2\right]\right)\tag{61}$$

is stationary, as the direct check

$$\partial_t\mathcal{P}=\int dxdy\left[\frac{1}{2}\lambda_{uu}(\boldsymbol{\nabla}\cdot\mathbf{u})\nu_h\nabla^2 h+\frac{\nu_h}{\nu_u}\lambda_{uu}\boldsymbol{\nabla}h\cdot(\boldsymbol{\nabla}\cdot\mathbf{u})\nu_u\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\right]\tag{62}$$

$$=-\int dxdy\,\frac{\nu_h\lambda_{uu}}{2}\boldsymbol{\nabla}\cdot[\boldsymbol{\nabla}h(\boldsymbol{\nabla}\cdot\mathbf{u})^2]\tag{63}$$

$$=0,\tag{64}$$

confirms. In terms of the density fluctuations

$$\rho\equiv\boldsymbol{\nabla}\cdot u\tag{65}$$

and the vorticity

$$\Omega\equiv\partial_x u_y-\partial_y u_x,\tag{66}$$

we may interpret Eqs. (60) as describing a liquid that is flowing on a fluctuating surface $h(x,y,t)$ via the partially decoupled set of equations

$$\partial_t h=\nu_h\nabla^2 h+\frac{1}{2}\lambda_{uu}\rho^2+\eta_h,$$
$$\partial_t\rho=\nu_u\nabla^2\rho+\lambda_{hu}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}h)+\boldsymbol{\nabla}\cdot\boldsymbol{\eta},\tag{67}$$
$$\partial_t\Omega=\partial_x\eta_y-\partial_y\eta_x.$$

This is again reminiscent of the advection of a scalar [28,29] (density fluctuations are advected along height gradients) which is not quite passive, as ρ influences the evolution of h. Equations (67) describe also a special case of the coupled growth of a binary film or, equivalently, of a scalar (Ising) magnet [15].

With a vanishing bulk modulus $\nu_u^{(1)}+\nu_u^{(2)}=0$, prescriptions (I) and (II) imply

$$\lambda_{hh}=\lambda_{uu}^{(1)}=\lambda_{hu}^{(1)}=\lambda_{uu}^{(2)}+\lambda_{uu}^{(3)}=\lambda_{hu}^{(2)}+\lambda_{hu}^{(3)}=0\tag{68}$$

and

$$\nu_h\lambda_{uu}^{(2)}=\nu_u^{(1)}\lambda_{hu}^{(2)},\tag{69}$$

resulting in the equations of motion

$$\partial_t h=\nu_h\nabla^2 h+\frac{1}{2}\lambda_{uu}\partial_i u_j(\partial_i u_j-\partial_j u_i)+\eta_h,$$
$$\partial_t u_i=\nu_u\partial_j(\partial_j u_i-\partial_i u_j)+\lambda_{hu}\partial_j h(\partial_j u_i-\partial_i u_j)+\eta,\tag{70}$$

where $\nu_u\equiv\nu_u^{(1)}$, $\lambda_{uu}\equiv\lambda_{uu}^{(2)}$, and $\lambda_{hu}\equiv\lambda_{hu}^{(2)}$. Introducing a fictitious direction z according to the natural definition $\boldsymbol{\nabla}\times\mathbf{u}=(\partial_x u_y-\partial_y u_x)\hat{z}$ with $\boldsymbol{\nabla}=(\partial_x,\partial_y,0)$ and $\mathbf{u}=(u_x,u_y,0)$, we can rewrite Eqs. (70) in the more compact form

$$\partial_t h=\nu_h\nabla^2 h+\frac{1}{2}\lambda_{uu}(\boldsymbol{\nabla}\times\mathbf{u})^2+\eta_h,$$
$$\partial_t\mathbf{u}=-\nu_u\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u})-\lambda_{hu}\boldsymbol{\nabla}h\times(\boldsymbol{\nabla}\times\mathbf{u})+\boldsymbol{\eta}.\tag{71}$$

With this form at hand, the stationarity of the probability density

$$\mathcal{P}=N\exp\left(-\int dxdy\left[\frac{\nu_h}{2}(\boldsymbol{\nabla}h)^2+\frac{\nu_u}{2}(\boldsymbol{\nabla}\times\mathbf{u})^2\right]\right)\tag{72}$$

is obtained in direct analogy with the case of vanishing shear modulus. In terms of ρ and Ω, Eqs. (71) become

$$\partial_t h=\nu_h\nabla^2 h+\frac{1}{2}\lambda_{uu}\Omega^2+\eta_h,$$
$$\partial_t\Omega=\nu_u\nabla^2\Omega+\lambda_{hu}\boldsymbol{\nabla}\cdot(\Omega\boldsymbol{\nabla}h)+\eta_\perp,\tag{73}$$
$$\partial_t\rho=\eta_\parallel,$$

none other than Eqs. (67) with ρ and Ω interchanged, but still a transverse noise $\eta_\perp=\partial_x\eta_y-\partial_y\eta_x$ driving the fluctuations of Ω and a longitudinal noise $\eta_\parallel=\partial_x\eta_x+\partial_y\eta_y$ driving ρ. Here we can interpret Ω as the density of dislocations whose presence locally affects growth.

Experiments on surface growth do not point to a unique characterization [30] and, in particular, most measured roughness exponents differ from those expected on the basis of the KPZ equation [8,31,32]. This may result from conservation laws, incorporated in some molecular beam epitaxy models [33–36], but it also may be the consequence of the dynamic coupling of the height fluctuations with the fluctuations of the intrinsic order parameter of the material, as in magnetic or crystalline growth. Thus, having established that the special Eqs. (60) and (71) admit Gaussian steady states, one would like to know whether the full Eqs. (56) may flow to them under renormalization. If Eqs. (60) or (71) indeed have a basin of attraction, we are left with the surprising conclusion that the coupling to crystal vibration tethers the fluctuating surface, in an appropriately prepared sample, to logarithmic roughness, in marked contrast to the KPZ roughness associated with amorphous growth.

Rather than renormalizing the full set of Eqs. (56), one may, as a first attempt, consider the renormalization of Eqs. (67) or (73) with an added KPZ term ($\lambda_{hh}\neq 0$),

$$\partial_t h = \nu_h \nabla^2 h + \frac{1}{2}\lambda_{hh}(\nabla h)^2 + \frac{1}{2}\lambda_{uu}\rho^2 + \eta_h,$$
$$\partial_t \rho = \nu_u \nabla^2 \rho + \lambda_{hu} \nabla\cdot(\rho \nabla h) + \eta_\parallel, \quad (74)$$
$$\partial_t \Omega = \eta_\perp .$$

The subspace of Eqs. (74) is closed under renormalization since they are invariant under the transformation

$$\mathbf{u} \to \mathbf{u} + \nabla \times \mathbf{A}, \quad (75)$$

with $\mathbf{A}(x,y)$ an arbitrary smooth function. (Similarly, Eqs. (73) are invariant under

$$\mathbf{u} \to \mathbf{u} + \nabla \phi, \quad (76)$$

with ϕ arbitrary.) Also, Eqs. (74) are interesting in their own right: the field ρ may be interpreted as density fluctuations, e.g., of surfactants, sliding on the surface. Similarly, if h describes the height of a material surface (such as a liquid film), ρ may play the role of material surface density fluctuations (proportional to the thickness of the film in the case of an incompressible liquid). These density fluctuations are conserved as the particles, or liquid elements, slide along gradients in the surface height, according to Eqs. (74). However, the implicit $\rho \to -\rho$ symmetry of Eqs. (74) is difficult to justify physically for surfactants or liquid films.

As a final nonperturbative observation on Eq. (74), we note that they are invariant under the infinitesimal tilt

$$\mathbf{x} \to \mathbf{x} + \lambda \boldsymbol{\epsilon} t,$$
$$h \to h + \boldsymbol{\epsilon}\cdot\mathbf{x} \quad (77)$$

in the subspace $\lambda_{hh} = \lambda_{hu} \equiv \lambda$, so that λ_{hh} and λ_{hu} are not modified by coarse graining. Their renormalization flows are then given by

$$\frac{\partial \lambda_{hh}}{\partial l} = (z_h + \zeta_h - 2)\lambda_{hh},$$
$$\frac{\partial \lambda_{hu}}{\partial l} = (z_\rho + \zeta_h - 2)\lambda_{hu}, \quad (78)$$

where z_h and z_ρ are the dynamical exponents associated to the fields h and ρ, respectively, and ζ_h is the height roughness exponent. If a fixed point occurs at $\lambda_{hh} = \lambda_{hu} \neq 0$, the relation

$$z_h + \zeta_h = z_\rho + \zeta_h = 2 \quad (79)$$

holds exactly, while stability of a fixed point with $\lambda_{hh} = \lambda_{hu} = 0$ requires

$$z_h + \zeta_h \leqslant 2 \quad \text{and} \quad z_\rho + \zeta_h \leqslant 2. \quad (80)$$

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APPENDIX A: NON-GAUSSIAN STEADY STATES

In this appendix, we show through a simple example how the method used in the bulk of the paper may be extended to search for non-Gaussian steady states. We focus on equations of the form of Eq. (17) in Sec. III, where the usual Laplacian smoothing term is promoted to a more general object.

In parallel, we promote $\mathcal{Q}$ [see Eq. (23)] to include higher order terms, as

$$\mathcal{Q} = \int d^d x (\nu_{ij\alpha\beta}\partial_\alpha h_i \partial_\beta h_j + \pi_{ijk\alpha\beta\gamma}\partial_\alpha h_i \partial_\beta h_j \partial_\gamma h_k + \rho_{ijkl\alpha\beta\gamma\delta}\partial_\alpha h_i \partial_\beta h_j \partial_\gamma h_k \partial_\delta h_l + \cdots), \quad (A1)$$

while still imposing the relation of Eq. (25) between $\mathcal{L}$ and $\mathcal{Q}$. Following the procedure of Sec. IV in the special case of a single field and in two dimensions (for the sake of simplicity), we obtain the additional prescriptions

$$\pi_{xx\alpha}\lambda_{yy} - 2\pi_{xy\alpha}\lambda_{xy} + \pi_{yy\alpha}\lambda_{xx} = 0 \quad (A2)$$

for any α, and

$$\rho_{xx\alpha\beta}\lambda_{yy} - 2\rho_{xy\alpha\beta}\lambda_{xy} + \rho_{yy\alpha\beta}\lambda_{xx} = 0 \quad (A3)$$

for any α, β. Here π and ρ have been symmetrized, and α, β denote x or y.

Let us focus on the quartic term with coefficient ρ. For rotationally symmetric tensors

$$\lambda_{\alpha\beta} = \lambda\,\delta_{\alpha\beta} \quad (A4)$$

and

$$\rho_{\alpha\beta\gamma\delta} = \rho(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}), \quad (A5)$$

Eq. (A3) has only the trivial solution $\lambda = 0$. By contrast, an antisymmetric nonlinear term with

$$\lambda_{xx} = -\lambda_{yy} \equiv \lambda \quad \text{and} \quad \lambda_{xy} = 0, \quad (A6)$$

allows the nontrivial solution

$$\rho_{xxxx}=\rho_{xxyy}=\rho_{yyyy}\equiv\rho_1,\qquad \rho_{xxxy}=\rho_{yyyx}\equiv\rho_2/2, \tag{A7}$$

with arbitrary ρ_1 and ρ_2. Thus, the equation of evolution

$$\partial_t h=\nu\nabla^2 h+\rho_1[(\nabla h)^2\nabla^2 h+\partial_x h\partial_y h\partial_x\partial_y h]+\rho_2[\partial_x h\partial_y h\nabla^2 h+(\nabla h)^2\partial_x\partial_y h]+\tfrac{1}{2}\lambda[(\partial_x h)^2-(\partial_y h)^2]+\eta, \tag{A8}$$

obtains the non-Gaussian steady-state distribution

$$\mathcal{P}_{\rm quartic}=N\exp\left(-\int dxdy\left\{\frac{\nu}{2}(\nabla h)^2+\frac{\rho_1}{12}[(\partial_x h)^4+6(\partial_x h)^2(\partial_y h)^2+(\partial_y h)^4]+\frac{\rho_2}{6}[(\partial_x h)^3\partial_y h+(\partial_y h)^3\partial_x h]\right\}\right). \tag{A9}$$

We note that this equation of evolution, too, satisfies the hidden symmetry of Eq. (16).

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